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# SOLAR TACHOCLINE REVISITED

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## ABSTRACT

Using recent helioseismological data, Kosovichev has shown that half of the tachocline lies within the convective zone (CZ). Previous theoretical models suggested that it lies outside the CZ. We propose a new model whereby the tachocline originates within the CZ and/or in that part of the overshooting region where the convective flux is still positive. The key ingredients of the model are shear, vorticity, and buoyancy. We find that (1) shear alone, (2) shear + vorticity, and (3) shear + buoyancy are unable to reproduce the measured Reynolds stresses at the surface of the Sun. The key ingredients are *vorticity + buoyancy*, both of which are missing in previous models. Without carrying out a detailed numerical calculation, we estimate the thickness of the tachocline to be 0.053 (in units of the solar radius) compared with Kosovichev's value of $0.09 \pm 0.04$. The next step is the numerical solution of the equations.

*Subject headings:* convection — Sun: interior — Sun: rotation — turbulence

## 1. INTRODUCTION

Helioseismological data (Thompson et al. 1996) show that a surface-like differential rotation $\tilde{\Omega}(\theta)$ persists through the entire convective zone (CZ) and that near the lower base of it, there is a transition to solid-body rotation, $\tilde{\Omega}$ = constant. The region of rapid change, the tachocline, has a thickness $h$ that Kosovichev (1996) has shown to be (in units of solar radius)

$$h = 0.09 \pm 0.04. \tag{1a}$$

The same data also show that the endpoint of the CZ adiabatic stratification is located at

$$0.713 \pm 0.003. \tag{1b}$$

Thus, the midpoint of the tachocline is located only slightly below the convective zone, at

$$0.692 \pm 0.005. \tag{1c}$$

Half of the tachocline is therefore within the CZ.

Spiegel & Zahn (1992, hereafter SZ) assumed that the tachocline lies in the stably stratified region below the CZ. The new result (eq. [1c]), together with the accepted notion that differential rotation and buoyancy are intimately related and the fact that differential rotation gets smoothed out where convection also begins to weaken, makes it somewhat difficult to visualize a tachocline that occurs independently of the CZ.

We suggest an alternative model whereby the tachocline originates where convection is still active: this includes the CZ proper as well as the overshooting region where $\nabla - \nabla_{\rm ad} < 0$ but the convective flux is positive (Canuto 1997b). The new model differs from previous models in other important aspects. These models make three basic assumptions: (1) the tachocline lies below the CZ in the stably stratified region, (2) the source of turbulence (Reynolds stresses $\tau_{ij}$) is the differential rotation itself, and (3) the horizontal viscosity $\nu_H \gg \nu_V$ (vertical viscosity). The sequence of events can be formally represented as

$$\tilde{\Omega}(\theta) \rightarrow \tau_{ij} \rightarrow \tilde{\Omega}(\theta), \tag{1d}$$

and such models can only claim self-consistency, a bootstrap approach. It is often stated that assumption 3 parallels that in oceanic turbulence, creating the impression that one has a strong basis to rely on. It is important to stress that this is not quite so. In the ocean's first $\sim 10^2$ m (the mixed layer [ML], broadly equivalent to the stellar CZ), turbulence is created by wind stresses, and one can reliably compute $\nu_V$; however, $\nu_H$ is not computed but treated as a free parameter. Below the ML, turbulence subsides considerably (up to a factor of $\sim 10^3$), and this is due to a variety of sources, primarily internal waves. Contrary to the stellar case, the spectrum of the ocean internal waves is known (Gargett et al. 1981), yet there is no agreement on even the functional form of $\nu_V = c\epsilon N^{-2}$ ($N > 0$ is the Brunt-Väisälä frequency, $\epsilon$ is the rate of dissipation of the turbulent kinetic energy $K$) as various forms have been proposed: $\epsilon \sim KN$, $\epsilon \sim KN^{3/2}$, and $\epsilon \sim (KN)^2$ (Gargett & Holloway 1984; Moum 1996; Gregg 1989; Gargett 1990). In conclusion, in the ocean case, in spite of a considerably rich trove of experimental information about waves, dissipation rates, etc., the form of $\nu_V$ below the ML is still an open question, while $\nu_H$ is treated as an adjustable quantity. It thus seems hardly justified to use ocean turbulence as a template for stellar turbulence, where one has less data to check the consistency of one's assumptions. To give a concrete example, consider the SZ expression for $h$:

$$h = 0.03 \left(\frac{\chi}{\nu_H}\right)^{1/4}. \tag{1e}$$

Since, at the base of the CZ, $\chi = 2 \times 10^7$ cm$^2$ s$^{-1}$ (Elliott 1997), in order to fit the observed value (eq. [1a]),

$$\nu_H = 2 \times 10^5 \text{ cm s}^{-1}. \tag{1f}$$

The problem is not that of matching equation (1a), which is trivially done with one free parameter in equation (1e), but to show what equation (1f) implies. The Reynolds stresses $\tau_{ij}$ are usually assumed to be related to the "mean flow" field via the relation $\boldsymbol{\tau} \sim \boldsymbol{S}$ or, specifically,

$$\tau_{\theta\phi} = -\nu_H \sin\theta \frac{\partial\tilde{\Omega}}{\partial\theta}\,, \ \tau_{r\phi} = -\nu_V r \sin\theta \frac{\partial\tilde{\Omega}}{\partial r}\,, \tag{1g}$$

with $\nu_V \neq \nu_H$. What do equations (1f) and (1g) entail? We rewrite the $\boldsymbol{\tau}$ versus $\boldsymbol{S}$ relation as

$$K^{-1}\boldsymbol{\tau} = (\nu_H K^{-1} S)\,(\boldsymbol{S}/S)\,. \tag{2a}$$

Since $K^{-1}\boldsymbol{\tau}$ and $\boldsymbol{S}/S$ are of $\sim O(1)$, it follows that $\nu_H \sim KS^{-1} \sim K\tilde{\Omega}^{-1}$. Since we can assume that production equals dissipation ($P = \epsilon$), with $P = -\{\boldsymbol{\tau S}\} \sim \nu_H \tilde{\Omega}^2$, we have a second relation $\epsilon = \nu_H \tilde{\Omega}^2$. Finally, since, in the presence of strong stratification, the energy spectrum $E(k) \sim (\epsilon N)^{1/2} k^{-2}$ rather than the Kolmogorov regime, which holds at much higher $k$'s, the resulting kinetic energy is $K \sim (\epsilon N)^{1/2} l$, where $l$ is the size of the largest eddy. Putting these three relations together, we derive $[N^2 = gH_p^{-1}(\nabla - \nabla_{\rm ad})]$

$$\nu_H \sim l^2 N,\ l \sim 0.1\ \text{km} \sim 10^{-5} h. \tag{2b}$$

Is this $l$ physically meaningful? Since we are unable to answer the question, we are equally unable to assess the overall internal consistency of the model. Finally, when equations (1g) are applied to the solar surface, they do not fit the data (Gilman & Howard 1984) (see Fig. 1).

## 2. NEW MODEL

The goals of the new model can be summarized as follows: (*a*) test the new model in a region other than the tachocline itself, (*b*) identify the source of turbulence so as to avoid a bootstrap approach, (*c*) try to reproduce equaton (1a), (*d*) avoid the use of adjustable parameters, and (*e*) understand physically the consequences of the model (e.g., what is the new relation [2b]?). Since we work within the CZ, we consider stratification and a mean flow. Thus,

$$\tau_{ij} = \tau\,(S_{ij}, V_{ij}, h_i)\,,\ h_i = h\,(\beta_i, \tau_{ij})\,, \tag{3a}$$

where $S_{ij}$, $V_{ij}$, $\beta_i$, and $h_i$ are the shear, vorticity, temperature gradient, and convective flux $h_i \equiv \overline{u_i' T'}$. The explicit forms of equations (3a) can be found in Canuto, Minotti, & Schilling (1994). To accomplish (*a*), we have solved equations (3a) using only the partial combinations:

$$\text{Shear, Shear + Vorticity, Shear + Buoyancy.} \tag{3b}$$

None of them reproduce the data (Fig. 1). The key ingredient turns out to be the interaction

$$\text{Buoyancy + Vorticity.} \tag{3c}$$

Thus, instead of equation (1d), we suggest the following chain of events:

$$\text{Buoyancy} \rightarrow \tau_{ij} \rightarrow \tilde{\Omega}(\theta). \tag{3d}$$

The source of turbulence is buoyancy, which naturally generates $\tau_{ij}$ (which in turn generates differential rotation through the interaction [3d]). We have accomplished (*a*) and (*b*) above.

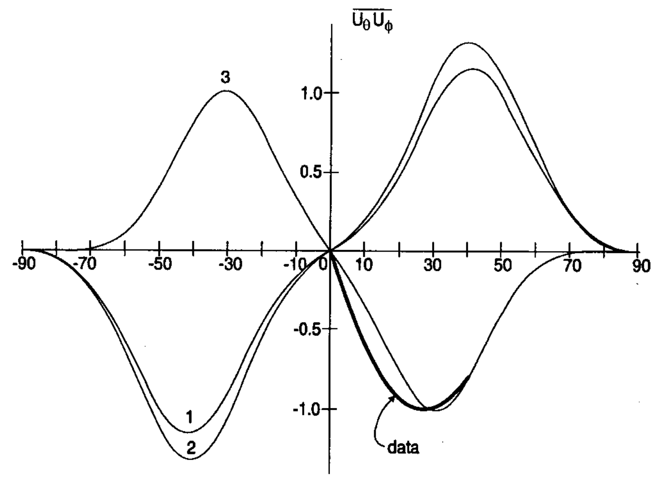


FIG. 1.—The Reynolds stresses $\overline{u_\theta u_\phi}$ measured at the surface of the Sun vs. (co)latitude. Curve 1 represents the first two combinations in eqs. (3b), i.e., the first of eqs. (1g). They give almost the same result, which clearly does not fit the data. Curve 2 represents the third combination in eqs. (3b), which is equally unsatisfactory. Only the combination (3c) is able to reproduce the data, curve 3.

To accomplish (*c*), we should substitute the full form of $\tau_{ij}$ given by equations (3a), with all the constants computed from within the same model, into the angular momentum equation

$$r\sin\theta \frac{\partial}{\partial t} v_\phi + 2r\sin\theta\,(\Omega x v)_\phi = -\frac{\sin\theta}{r^2}\frac{\partial}{\partial r}\Lambda_{r\phi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Lambda_{\theta\phi},$$

$$\Lambda_{r\phi} \equiv r^3\,(\tau_{r\phi} + v_r v_\phi)\,,\ \Lambda_{\theta\phi} = \sin^2\theta\,(\tau_{\theta\phi} + v_\theta v_\phi)\,. \tag{4a}$$

Here $v_r$, $v_\theta$, and $v_\phi = r\tilde{\Omega}\sin\theta$ are the meridional-azimuthal components of $v$. We shall assume that the main result of the SZ model, equation (1e), remains valid provided one changes $\chi \rightarrow \chi + \chi^{rr}$, where $\chi^{rr}$ is the radial turbulent conductivity entering the convective flux ($\beta$ is the superadiabatic temperature gradient),

$$\overline{u_r' T'} = \chi^{rr} \beta_r. \tag{4b}$$

The equation for the convective flux is given by equations (38a) and (38b) of Canuto et al. (1994). As shown in that paper, the solutions $\tau_{\theta\phi}$ and $\tau_{r\phi}$ of equations (3a) are

$$\tau_{\theta\phi} = F_1\tilde{\Omega} + F_2 \sin\theta \frac{\partial\tilde{\Omega}}{\partial\theta} + F_3 \sin\theta r \frac{\partial\tilde{\Omega}}{\partial r}\,, \tag{4c}$$

$$\tau_{r\phi} = G_1\tilde{\Omega} + G_2 \sin\theta \frac{\partial\tilde{\Omega}}{\partial\theta} + G_3 \sin\theta r \frac{\partial\tilde{\Omega}}{\partial r}\,, \tag{4d}$$

where the functions $F$ and $G$ (cm$^{2\ -1}$) are given by equations (53a)–(54e) of Canuto et al. (1994). Clearly, $-F_2$ and $-G_3$ correspond to the $\nu_H$ and $\nu_V$ in the SZ model. Using the numerical results of Figures 4–8 (Canuto et al. 1994), we find, at a colatitude of 30°,

$$10^2\epsilon = 4\epsilon_*,\ 10^2K = 4e_*,\ \frac{1}{2}\overline{u_r'^2} = 210^{-2}e_*,$$

$$\frac{1}{2}\overline{u_\theta'^2} = 810^{-3}e_*,\ \overline{u_r'u_\theta'} = 610^{-3}e_*,\ 10^2\chi^{rr} = 4e_*\Omega_*^{-1}, \quad (4e)$$

where $e_* = \frac{1}{2}\epsilon_*\Omega_*^{-1}$. Using Canuto et al.'s equation (53c) for $\nu_H$, we find

$$\chi^{rr} = 10\nu_H,\ h = 0.053. \quad (4f)$$

Considering that we have not carried out a numerical computation of the full model, the value of $h$ compares favorably with equation (1a). It is important to stress that neither $\nu_H$ nor $\chi^{rr}$ were chosen to fit equation (1a); rather, they were determined from within the model. This satisfies criterion *d*. If we substitute $\nu_H$ into equations (2b), we obtain

$$l \sim 10^3\ \text{km} \sim 0.1h. \quad (4g)$$

Equation (4g) seems physically more correct than equations (2b). We also note that our $\nu_H$ is of the same order as that obtained, for example, by Durney (1991). Two more considerations are in order concerning the key role played by vorticity $V_{ij}$ in equations (3a). First, since

$$V_{r\phi} = -2\tilde{\Omega}\sin\theta - S_{r\phi},\ V_{\theta\phi} = -2\tilde{\Omega}\cos\theta - S_{\theta\phi}, \quad (5a)$$

$V_{ij}$ introduces a novel feature: a term that depends on $\tilde{\Omega}$ itself rather than on its derivatives. It was originally suggested on empirical grounds (Biermann 1951; Rüdiger 1989), but here it originates naturally when allowance is made for vorticity. Second, within a standard turbulence model, to obtain $\nu_H \gg \nu_V$, one must include vorticity. In tensorial form, the functional form of equations (3a) is

$$\boldsymbol{\tau} = -\nu_T\boldsymbol{S} + \boldsymbol{B} + (\boldsymbol{\tau V} + \boldsymbol{V\tau})\,. \quad (5b)$$

Using a perturbative approach beginning with $\boldsymbol{\tau}^0 \sim \boldsymbol{S}$, one easily obtains from equation (5b)

$$\boldsymbol{\tau} = -g_1\boldsymbol{S} - g_2\tau(\boldsymbol{SV} - \boldsymbol{VS}) + \dots\ . \quad (5c)$$

We then have, using only the first term,

$$\tau_{xx} \sim g_1\frac{\partial U}{\partial x}\,,\ \tau_{xz} \sim g_1\frac{\partial U}{\partial z}\left[1 + \frac{\partial W}{\partial x}\left(\frac{\partial U}{\partial z}\right)^{-1}\right] \sim g_1\frac{\partial U}{\partial z}\,. \quad (5d)$$

Since $\partial U/\partial x \sim \partial W/\partial z$, or $UL_v \sim WL_h$ ($L_v$ and $L_h$ are the vertical and horizontal length scales with $L_h/L_v \gg 1$), the last relation in equations (5d) follows. Thus, the first term in equation (5c) can give only

$$\nu_H \sim \nu_V. \quad (5e)$$

Let us include vorticity. We now have

$$\tau_{xx} = -g_1\left[1 + g_3\tau\left(\frac{\partial U}{\partial x}\right)^{-1}\left(\frac{\partial U}{\partial z}\right)^2 + \dots\right]\frac{\partial U}{\partial x}\,, \quad (5f)$$

which yields the horizontal diffusivity

$$\nu_H \approx 1 + g_3\frac{\tau U^2}{WL_v}\,. \quad (5g)$$

Similarly, one derives

$$\tau_{xz} = -\frac{1}{2}g_1\left(1 + g_4\tau\frac{\partial U}{\partial x}\right)\frac{\partial U}{\partial z}\,,\ \nu_V \sim 1 + g_4\tau\frac{\partial U}{\partial x}\,. \quad (5h)$$

Thus, finally,

$$\frac{\nu_H}{\nu_V} \approx \left(\frac{L_h}{L_v}\right)^2 \gg 1. \quad (5i)$$

Thus, for $\nu_H \gg \nu_V$, one needs vorticity, a conclusion in agreement with equation (3c), which requires vorticity on independent grounds. The model is self-consistent.

## 3. NEW MODEL: MEAN TEMPERATURE EQUATION

The full model is made of equations (4a), the expressions for $\tau_{ij}$ and $h_i$, and the equation for the mean $T$, which is usually a simplified version of the complete equation (Canuto 1997a):

$$\rho\frac{D}{Dt}(c_pT + K + K_v + G)$$
$$= -\frac{\partial}{\partial x_i}(F_i^r + F_i^c + F_i^{\text{ke}} + \rho\tau_{ij}v_j) - \frac{\partial p}{\partial t}\,. \quad (6a)$$

Here $D/Dt \equiv \partial/\partial t + v_j\partial/\partial x_j$, and since the gravitational field $G$ does not depend on time, $DG/Dt = g_iv_i$; $K$ and $K_v$ are the kinetic energies of the turbulent field $K = \frac{1}{2}\tau_{ii}$ and of the $v$-field. In the right-hand side of equation (6a), we have the radiative, convective, and turbulent kinetic energy ($F_i^{\text{ke}} = \frac{1}{2}\rho u_k'u_k'u_i'$) fluxes, plus a new term representing the *transport of the Reynolds stresses by the large-scale velocity* $v$*-field*. Equation (6a) is the generalized Bernoulli equation that includes turbulence and radiation. Equation (2.11) of SZ corresponds to equation (6a) with

$$c_pT \gg (K, K_v, G)\,,\ F_r^i \gg (F_c^i, F_i^{\text{ke}}, \tau_{ij}v_j)\,,\ \frac{\partial p}{\partial t} = 0. \quad (6b)$$

The new term $v_j\tau_{ij}$ in equation (6a) gives rise to terms of the form $a_ri_r + a_\theta i_\theta + a_\phi i_\phi$ with $a_r = 2u_\phi\tau_{r\phi}$, $a_\theta = 2u_\phi\tau_{\theta\phi}$, and $a_\phi = u_\phi\tau_{\phi\phi}$. Thus, in the radial component, the temperature equation will be supplemented by the term $2\rho u_\phi\tau_{r\phi} = 2\rho r\sin\theta\tilde{\Omega}\tau_{r\phi}$.

## 4. NEW MODEL: REYNOLDS STRESSES, CONVECTIVE FLUXES

The expressions for $\tau_{ij}$ and $h_i$ in equations (3a) are given by equations (37)–(38b) of Canuto et al. (1994). The turbulent kinetic energy and the dissipation rate $\epsilon$ are given by two differential equations, equation (39a) and equation (44a) from Canuto et al. One may make the equation for $K$ algebraic, but the differential equation for $\epsilon$ should be kept because local approximations introduce an undetermined mixing length.

## 5. CONCLUSIONS

Given the complexity of the problem, it is not surprising that we have not yet reached a satisfactory picture of the tachocline. SZ suggested the first model, but it may be difficult to reconcile the picture of a tachocline outside the CZ with the new data showing that half of it lies within the CZ. Furthermore, if the tachocline lies below the CZ, the turbulent $\tau_{ij}$ can no longer be generated by convection; the burden then shifts to explain the origin of $\tau_{ij}$ and a viscosity $\nu_H \sim 10^3$ smaller (Elliott 1997) than what the model yields (Zahn 1992).

We suggest that the tachocline may originate within the CZ and/or in that part of the overshooting region where $\nabla - \nabla_{\rm ad} < 0$ but the convective flux is still positive (Canuto 1997b). This seems to alleviate several problems: the origin of turbulence is no longer an issue since buoyancy generates $\tau_{ij}$; through the buoyancy-vorticity interaction, the $\tau_{ij}$ then generate the $\tilde{\Omega}(\theta)$; one no longer has to "choose" a $\nu_H$ to fit equation (1a) since the value is determined from within the model itself; the resulting thickness $h$ is quite acceptable even without a full solution of the model. From a physical viewpoint, linking differential rotation with convection seems quite natural for a variety of reasons: (i) we know from helioseismological data that the differential rotation gets smoothed out where convection itself begins to weaken, and (ii) while shear goes abruptly to zero when $\tilde{\Omega}(\theta) \rightarrow \tilde{\Omega} =$ constant, vorticity goes smoothly into the rigid-body rotation; joining the two regions thus seems more naturally accomplished with vorticity than with shear. Finally, the main ingredients of the new model have been successfully tested against measured data for both $\tau_{\theta\phi}$ and $\tilde{\Omega}(\theta)$ at the surface of the Sun. The value of the tachocline thickness $h$ that we have determined should be viewed only as a first justification of the consistency of the model. The next step consists of solving the new equations for $\tilde{\Omega}$, $T$, $\tau_{ij}$, and $h_i$.

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